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MODIFIED GRAEFFE'S ROOT SQUARING METHOD WITH SOLVABILITY CONDITIONS

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Abstract- It is been said that Graeffe's method determines all the roots of an algebraic equation real and complex, repeated and non-repeated simultaneously. In this study, it is said that this statement is not universally true. It has been shown [1, 2] that the method is valid if the algebraic equations satisfy the conditions- (i) equations with zero-coefficient must have at least one pair of equidistant non-zero coefficient from the zero–coefficient; (ii) any transformed equation of a given equation with non-zero coefficient may have zero coefficients but first condition (i) must satisfy these new coefficients and (iii) all the coefficients of non-linear algebraic equation must not be unity. GRAEFFE.BAS program [2] has been modified in these lights and has been extended for not solvable equations.

Keywords— Graeffe's method, root squaring, zero coefficient, equidistant, solvability conditions, transformed equation, non-linear algebraic equation.

1 INTRODUCTION

Non-linear algebraic equations arise as (i) auxiliary equation of ODE and Difference equations, (ii) characteristics equation of matrix eigen value problem. Graeffe's root squaring method for solving non-linear algebraic equations is a well known classical method. It was developed by C. H. Graeffe in 1837. Its explanation, uses and advantages are available in many treatises and literatures. Hutchinson [3] described the method to be very useful in aerodynamics and in electrical analysis. He put forward a list of all rules for the identification of the nature of the roots and stated the advantages of the method. Cronvich [4] gave another new set of such rules and compared these rules with those of Hutchinson. He recommended these methods of approximating irrational and imaginary roots. The paper of Hutchinson [3] and Cronvich [4] contains a good number of references and review literatures on these subjects. Bodewing [5] also pointed out the advantages of the method in finding all roots simultaneously. He also called the method best and showed the quadratic convergence of errors in each subsequent transformed new equation. Kopal [6] illustrate the method as the best way of extracting complex roots. Scarborough [7] said, "Probably the root squaring method of Graeffe is the best to use in "most cases". This method gives all the roots at once, both real and complex. But he did not mention the "cases". Carnahan et al [8] emphatically remarked the method as the most completely "global" one in the sense of yielding simultaneous approximation to all roots. He also designed a FORTRAN program to evaluate the distinct roots. Constantinles [2] called the method universal in its ability to locate all roots of polynomial equations. He also designed a personal computer program in BASIC without mentioning any exception. This program finds only two complex roots. Balagurusamy [9] also mentioned their advantage that is requires no initial guess but the coefficients must be real.

The purpose of this paper is to show that the foreside claims cannot always be accepted. It will be shown that there are limitations in applying Graeffe's root squaring method for solving algebraic equations although it finds all kinds of roots real, equal and imaginary.

2 OBSERVATIONS

Some of the researchers in the references discuss the following observations. Their comments about the method regarding advantages and disadvantages reflect their points of view.

(i) It is observed that there are equations, which are not transformable by root squaring into a different one with non-zero coefficients from where the roots of the equation are calculated. It is found that the odd degree equations set like

$$\left.\begin{array}{l} x^3+a=0,\ x^5+a=0,\ x^7+a=0, \\ x^5+2x^4+2x^3+2x^2+2x+1=0 \end{array}\right\} \quad (2.1)$$

etc. cannot be solved by the Graeffe's root squaring method manually as well as using GRAEFFE.BAS of Constantinides [2]. All these equations (2.1) transformed to the form $x^n+a=0$, $n=3,5,7$ after first iteration and GRAEFFE.BAS shows overflow at the statement 1080 of Constantinides [2]. It is observed that

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$$\left.\begin{aligned} &x^2 + x + b = 0, \quad x^4 + a = 0, \quad x^7 + ax^2 + b = 0, \\ &x^7 + ax^6 + b = 0, \quad x^4 + 2x^3 + 2x^2 + 2x + 2 = 0 \end{aligned}\right\} \quad (2.2)$$

etc. can be solved by both manual procedures and using program of Constantinides [2]. In this case zero coefficient revive and non zero coefficients do not vanish. As a result final transform equation in each case provides solution. Householder [10] passes a similar remark for $x^n - 1 = 0$ in which all n roots have unit modulus and the Graeffe's method fails for such equations without deriving any solvability condition. Similarly, Wilkinson [11] called for equation $x^2 - 1 = 0$ well-condition zeros $x = \pm 1$, because on squaring the transformed equation becomes $x^2 - 2x + 1 = 0$ and deteriorates to coincident ill-conditioned root 1.

(ii) Actual computation reveals another fact that 4th, 6th, 8th, 10th etc. degree non-linear equations with positive unit coefficients like $x^4 + x^3 + x^2 + x + 1 = 0$ will never stop the procedure because the coefficients of the transformed equation remain unity i.e. $(-1)^4 \; f(\sqrt{x}) \;\; f(-\sqrt{x})$ remains invariant.

3 Solvability Conditions

The observation in sec. 2 leads condition of solvability. In Gareffe's method, roots of the original equation are obtained from the coefficients of the last transformed equation. If some of those intermediate coefficients between the first and last one are zero, the roots may be infinite or even indeterminate. So, the first and foremost criteria of Graeffe's root squaring method to be successful is that the coefficients of the last transformed equation must be non-zero which in turn depends on some or all of the non-zero coefficients of the original equation.

Let the nth degree algebraic equation be

$$a_0 x^n + a_1 x^{n-1} + a_2 \; x^{n-2} + \ldots\ldots + a_{n-1} \; x + a_n = 0, \qquad a_0 \neq 0$$

The scheme of the process is as follows:

variable	x^n	x^{n-1}	x^{n-2}	$\cdots$	x^k	$\cdots$	x	1
	+	-	+		$(-1)^k$		$(-1)^{n-1}$	$(-1)^n$
coefficients	a_0	a_1	a_2		a_k		a_{n-1}	a_n
	a_0^2	$-a_1^2$	$+a_2^2$	$\cdots\cdots$	$(-1)^k a_k^2$		$\cdots(-1)^{n-1} a_{n-1}^2$	$(-1)^n a_n^2$
		$2a_0a_2$	$-2a_1a_3$	$\cdots\cdots$	$(-1)^{k+1} a_{k-1}a_{k+1}$		$\cdots(-1)^n 2a_{n-2}a_n$	
			$+2a_0a_4$	$\cdots\cdots$	$(-1)^{k+2} 2a_{k-2}a_{k+2}$	$\cdots$		
			$\cdots\cdots$		$(-1)^{k+l} 2a_{k-l}a_{k+l}$	$\cdots$		
			$\cdots\cdots$		$\cdots$			
2^{nd} power	${}_1A_0$	${}_1A_1$	${}_1A_2$	$\cdots\cdots$	${}_1A_k$	$\cdots\cdots$	${}_1A_{n-1}$ $\cdots\cdots$	${}_1A_n$
	$\vdots$	$\vdots$	$\vdots$	$\cdots\cdots$	$\vdots$	$\cdots\cdots$	$\vdots$ $\cdots\cdots$	$\vdots$
	$\vdots$	$\vdots$	$\vdots$	$\cdots\cdots$	$\vdots$	$\cdots\cdots$	$\vdots$ $\cdots\cdots$	$\vdots$
2^{j+1th} power	${}_{j+1}A_0$	${}_{j+1}A_1$	${}_{j+1}A_2$	$\cdots\cdots$	${}_{j+1}A_k$	$\cdots\cdots$	${}_{j+1}A_{n-1}$ $\cdots$	${}_{j+1}A_n$

where $\quad {}_{j+1}A_k = (-1)^k \left[{}_jA_k^2 + 2\sum_{l=1}^{k} (-1)^l . {}_jA_{k-l} . \, {}_jA_{k+l} \right], \quad 0 \le k \le n, \; k - l \ge 0, \; k + l \le n$

is a modification of what is given by Carnahan [8] and ${}_0A_k = a_k$. The equation suppose that the root squaring process stops after jth squaring according to the stopping conditions suggested in the scheme.

The conditions of solvability are discussed for the following two cases:

CASE A Equations with zero coefficients

Let $a_k = 0, \;\; 0 < k \le n$. If ${}_{j+1}A_k = 0$ for $0 < k \le n$ and for any j then the method will continue indefinitely without satisfying any stopping conditions suggested in the references and so the method fails. But if ${}_{j+1}A_k \neq 0$ at some stages of j then the method is successful in giving roots.

Let j=0, so

$${}_1A_k = (-1)^k \left[a_k^2 + 2\sum_{l=1}^{k} (-1)^l . a_{k-l} . a_{k+l} \right] \neq 0 \;\; \text{and} \;\; a_k = 0$$

$$2\sum_{l=1}^{k} (-1)^l . a_{k-l} . a_{k+l} \neq 0$$

i.e. $\quad a_{k-l} \neq 0 \;$ and $\; a_{k+l} \neq 0$.

i.e. from the zero coefficients $a_k = 0$, equidistant coefficients a_{k-l} and a_{k+l}, are non-zero.

Once ${}_1A_k$ become non-zero then the subsequent transformation will produce ${}_1A_k \neq 0$. This fact may be observed from the equation set (2.2) with zero coefficients in sec.2. But if it happens that ${}_{j+1}A_k = 0, \;\; j > 0$, then the solvability conditions follow the next case B.

CASE B Equation with non-zero coefficients

Let $a_k \neq 0, \;\; 0 < k \le n$. The case ${}_{j+1}A_k \neq 0, \, j > 0$ is of less interest of discussion. Because of such case the method is quite successful but if it happens that ${}_{j+1}A_k = 0, \, j > 0$ then the conditions will be same as the following.

Let ${}_{j+1}A_k = 0$, $0 < k \le n$, also $j = 0$.

Then $\; {}_1A_k = (-1)^k \left[a_k^2 + 2\sum_{l=1}^{k} (-1)^l . a_{k-l} . a_{k+l} \right] = 0$

$$\therefore \; (-1)^k a_k^2 + 2\sum_{l=1}^{k} (-1)^{l+k} . a_{k-l} . a_{k+l} = 0$$

i.e. the coefficients of the second powers of the roots in the transformed equation are zero. If these coefficients satisfy, the conditions like CASE A then the given equation is solvable under this method otherwise not. Similarly in any stages of transformation ${}_{j+1}A_k = 0$ but satisfy the conditions like CASE A then the given equation is solvable. Consider the equation

$$x^5 + 2x^4 + 2x^3 + 2x^2 + 2x + 1 = 0$$

in set (2.1) for which ${}_1A_k = 0, \;\; 0 < k < 5$ and which do not satisfy the criteria in CASE A. So, for this equation Graeffe's method fails whereas the method is successful for the equation

$$x^4 + 2x^3 + 2x^2 + 2x + 2 = 0 .$$

4 Modification of GRAEFFE.BAS

It is observed that there are equations, which are not transformable by root squaring into a new

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one with non-zero coefficients from where the roots of the original equation are calculated. It is found that the equations like

$$\left.\begin{array}{l} x^3 + a = 0,\ x^5 + a = 0,\ x^7 + a = 0, \\ x^5 + 2x^4 + 2x^3 + 2x^2 + 2x + 1 = 0 \end{array}\right\} \quad (3.1)$$

etc. While running the problems with original program [2] shows "overflow" at statement

```
1080:  X (NR) = (A(R, K-1) /A(R, K)) ^ (1/M) * 10 ^
       ((FACT(R, K-1) - FACT(R, K)) /M).
```

Overcoming ways of this difficulty have been discussed in [12]. For this purpose the original program of Constantinides [2] needs modification and extension. Therefore, it should be introduced the following statements and to generate a new subroutine 7. Statement numbers in Constantinides [2] are 10, 20, 30, and 6530. Modify statement numbers are set in-between. The following modify statements numbers are different from the original numbers. The new subroutine 7 is the extension part of the title of this paper which follows next.

```
603:  IF ABS (A(R, I)) = 0 THEN GOTO 605 ELSE GOTO 610
605:  IF ABS (A(R, I-1)) <> 0 AND ABS (A(R, I+1)) <> 0 THEN
      GOTO 1350
1075: IF C$(N-1) <> PS$ AND C$(1) <> PS$ THEN GOTO 1110
1350: GOSUB 7000
4065: IF C$(K) <> PS$ AND C$(K) <> NS$ THEN GOTO 4170
5085: IF C$(K) <> PS$ AND C$(K) <> NS$ THEN GOTO 5230

SUBROUTINE 7
7010: FOR I = N-1 TO 1 STEP -1
7020: IF ABS (A(R, I)) = 0 THEN GOTO 7030
7030: IF ABS (A(R, I-1)) <> 0 AND ABS (A(R, I+1)) <> 0 THEN
      GOTO 7050
7040: NEXT I
7050: FOR K = 1 TO N
7060: IF X (K) + XI (K) = 0 THEN GOTO 7090
7070: IF X (K) + XI (K) <> 0 THEN GOTO 1350
7080: NEXT K
7090: PRINT: PRINT "EQUATION IS NOT SOLVABLE BY
      GRAEFFE'S ROOT SQUARING METHOD"
7100:  ,
```

The difficulty of the problems (3.1) coefficients C$(N-1) and C$(1) do not show PURE SQUARE. If C$(1) do not show PURE SQUARE then from statement 1070 and statement 1110, we obtain K=0, which out of range of K at statement 1050. So we introduce new statement 1075 among them as follows:

```
1050:  FOR K = N TO 1 STEP -1
1070:  IF C$(K-1) <> PS$ THEN K = K-1:  GOTO 1110
1074:  New statement
1075:  IF C$(N-1) <> PS$ AND C$(1) <> PS$:  GOTO 1110
1110:  NEXT K
```

Also some of the problems (3.1) such as $x^3 + a = 0$ do not show PURESQUARE and NONSQUARE when K is N-1 to 1. In the original program it shows "overflow" at the statement 4070 and statement 5090. So we introduce statement 4065 and statement 5085 among them as follows:

```
4065:  IF C$(K) <> PS$ AND C$(K) <> NS$:  GOTO 4170
4070:  R (K) = (A(R, K-1) / A(R, K+1)) ^ (1/M) * 10 ^
       ((FACT(R, K-1) - FACT(R, K+1)) / M
5085:  IF C$(K) <> PS$ AND C$(K) <> NS$:  GOTO 5230
5090: R (K) =(ABS (A(R, K-1) /A(R, K+1)))^(1/M)*10
       ^((FACT(R, K-1) -FACT(R, K+1)) / M
```

The program in Constantinides [2] will be modified identifying the inability of solving some of the non-linear algebraic equations of the form (3.1). The modified GRAEFFE.BAS is developed using the following algorithm.

Algorithm

Identify (the presence of any zero coefficients)
or (all the coefficients are unity)
if (there is no pair of non zero coefficients from the zero coefficients)
or (all the coefficients are unity)
then go to print "equation is not solvable by Graeffe's root squaring method"
Repeat the above steps for each subsequent new transformed coefficient.

5. Numerical Examples

Some different numerical results are given below:

Example 1:

```
*********GRAEFFE'S ROOT-SQUARING METHOD********
****************** (GRAEFFE.BAS) ****************

DEGREE OF POLYNOMIAL        4
COEFFICIENT   4   IS      1
COEFFICIENT   3   IS      7
COEFFICIENT   2   IS      12
COEFFICIENT   1   IS     -4
COEFFICIENT   0   IS     -16
GIVE THE CONVERGENCE VALUE OF    F = 0.0001
ROOT SQUARING PROCESS
```

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R	A r 4	A r 3	A r 2	A r 1	A r 0
0	1.000E 0	7.000E0	1.200E1	-4.000E0	-1.600E0
1	10.000E-1	25.000E0	16.800E1	40.000E1	25.600E1
2	10.000E-1	28.900E1	87.360E2	73.984E3	65.536E3
3	10.000E-1	66.049E3	33.686E6	43.286E8	42.950E8
4	10.000E-1	42.951E8	56.296E13	18.447E18	18.447E18
5	10.000E-1	18.447E18	15.864E28	34.028E37	34.028E7
6	10.000E-1	34.028E37	12.555E57	11.579E76	11.579E6
7	10.000E-1	11.579E76	78.818E114	13.408E153	13.408E153
8	10.000E-1	13.408E153	31.072E230	17.977E307	17.977E307
9	10.000E-1	17.977E307	48.341E461	32.318E615	32.318E615
10	10.000E-1	32.316E615	11.749E924	0.445E%1232	

FACTOR EXCEEDS 999. POSIBILITY OF COMPLEX OR REPEATED ROOTS

THE COEFFICIENTS ARE
PURE SQUARE, PURE SQUARE, NON SQUARE, PURE SQUARE, PURE SQUARE,

THE NUMBER OF SQUARING: r = 9 THE POWER: m = 512

CALCULATION OF ROOTS:
FUNCTION WITH POSSITIVE VALUE OF (4) = 863.9999
FUNCTION WITH NEGATIVE VALUE IS (4) = -9.536743E-07
FUNCTION WITH POSSITIVE VALUE OF (1) = 4.768372E-07
FUNCTION WITH NEGATIVE VALUE IS (1) = -6
ONE PAIR OF COMPLEX OR REPEATED ROOTS:
CHECK FOR REPEATED ROOTS:
FUNCTION WITH POSSITIVE VALUE OF (2) = 96
FUNCTION WITH NEGATIVE VALUE IS (2) = 1.907349E-06
* ROOTS ARE REAL AND REPEATED AND OF THE SAME SIGN*
THE 4 ROOTS ARE:

-4
-2
-2
1

Example 2:

*********GRAEFFE'S ROOT-SQUARING METHOD***********

******************** (GRAEFFE.BAS) ******************

DEGREE OF POLYNOMIAL 4
COEFFICIENT 4 IS 1
COEFFICIENT 3 IS 0
COEFFICIENT 2 IS 0
COEFFICIENT 1 IS 0
COEFFICIENT 0 IS 1
GIVE THE CONVERGENCE VALUE OF F = 0.0001

ROOT SQUARING PROCESS

R	A r 4	A r 3	A r 2	A r 1	A r 0
0	1.000E0	0.000E0	0.000E0	0.000E0	0.000E0
1	10.000E-1	0.000E-1	60.000E-1	0.000E-1	10.000E-1
2	10.000E-1	-40.000E-1	60.000E-1	-40.000E-1	10.000E-1
3	10.000E-1	-40.000E-1	60.000E-1	40.000E-1	10.000E-1
4	10.000E-1	-40.000E-1	60.000E-1	40.000E-1	10.000E-1
5	10.000E-1	-40.000E-1	60.000E-1	40.000E-1	10.000E-1
6	10.000E-1	-40.000E-1	60.000E-1	40.000E-1	10.000E-1
7	10.000E-1	-40.000E-1	60.000E-1	40.000E-1	10.000E-1
8	10.000E-1	-40.000E-1	60.000E-1	40.000E-1	10.000E-1
9	10.000E-1	-40.000E1	60.000E-1	40.000E-1	10000E-1
10	10.000E-1	-40.00E-1	60.000E-1	40.000E-1	10.000E-1
11	10.000E-1	-40.000E-1	60.000E-1	40.000E-1	10.000E-1
12	10.000E-1	-40.000E-1	60.000E-1	40.000E-1	10.000E-1
13	10.000E-1	-40.000E-1	60.000E-1	40.000E-1	10.000E-1
14	10.000E-1	-40.000E-1	60.000E-1	40.000E-1	10.000E-1
15	10.000E-1	-40.000E-1	60.000E-1	40.000E-1	10.000E-1
16	10.000E-1	-40.000E-1	60.000E-1	40.000E-1	10.000E-1
17	10.000E-1	-40.000E-1	60.000E-1	40.000E-1	10.000E-1
18	10.004E-1	-40.000E-1	60.000E-1	40.000E-1	10.000E-1
19	10.000E-1	-40.000E-1	60.000E-1	40.000E-1	10.000E-1
20	10.000E-1	-40.000E-1	60.000E-1	40.000E-1	10.000E-1

ITERATIONS EXCEEDED. POSSIBILITY OF COMPLEX OR REPEATED ROOTS

THE COEFFICIENTS ARE:
PURE SQUARE, NON SQUARE, NON SQUARE, NON SQUARE, PURE SQUARE

THE NUMBER OF SQUARING: r=20
THE POWER: m= 1048576

CALCULATION OF ROOTS:
TWO PAIRS OF COMPLEX OR REPEATED ROOTS WITH IDENTICAL MODULI:
CHECK FOR REPEATED ROOTS
FUNCTION WITH POSSITIVE VALUE OF (1) = 2
FUNCTION WITH NEGATIVE VALUE IS (1) = 2
WARNING: CONVERGENCE NOT SATISFIED BY REAL ROOT
ROOTS ARE COMPLEX

*THE 4 ROOTS ARE:

0.7071 + 0.7071 i
0.7071 - 0.7071 i
-0.7071 + 0.7071 i
-0.7071 - 0.7071 i

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Example 3:

***********GRAEFFE'S ROOT-SQUARING METHOD**********

******************** (GRAEFFE.BAS) ******************

DEGREE OF POLYNOMIAL 3
COEFFICIENT 3 IS 1
COEFFICIENT 2 IS 0
COEFFICIENT 1 IS 0
COEFFICIENT 0 IS 1
GIVE THE CONVERGENCE VALUE OF F = 2

ROOT SQUARING PROCESS:

R	A_{r3}	A_{r2}	A_{r1}	A_{r0}
0	1.000E0	0.000E0	0.000E 0	1.000E0
1	10.000E-1	0.000E-1	0.000E -1	10.000E-1
2	10.000E-1	0.000E-3	0.000E -3	10.000E-1
3	10.000E-1	0.000E-7	0.000E-7	10.000E-1
4	10.000E-1	0.000E-15	0.000E-15	10.000E-1
5	10.000E-1	0.000E-31	0.000E-31	10.000E-1
6	10.000E-1	0.000E-63	0.000E-63	10.000E-1
7	10.000E-1	0.000E-127	0.000E-127	10.000E-1
8	10.000E-1	0.000E-255	0.000E-255	10.000E-1
9	10.000E-1	0.000E-511	0.000E-511	10.000E-1
10	10.000E-1	0.000E%-1023		

FACTOR EXCEEDS 999. POSSIBILITY OF COMPLEX OR REPEATED ROOTS

THE COEFFICIENTS ARE:
PURE SQUARE, , , PURE SQUARE

THE NUMBER OF SQUARING: r = 9 THE POWER: m = 512

CALCULATION OF ROOTS:
TWO PAIRS OF COMPLEX OR REPEATED ROOTS WITH IDENTICAL MODULI:
CHECK FOR REPEATED ROOTS:

THE 3 ROOTS ARE:
0
0
0
EQUATION IS NOT SOLVABLE BY GRAEFFE'S ROOT SQUARING METHOD

6 Conclusion

Literature survey of the past and the recent has been carried out. Classical method of Graeffe's root squaring method has been discussed. All problems not solvable manually and using GRAEFFE. BAS software in Loskor [12] has been discovered under observations. On the observation in sec. 2 the following solvability conditions of Graeffe's root squaring method has been derived in sec. 3.
(i) equation with zero-coefficient must gave at least one pair of equidistant non-zero coefficient from the zero-coefficient;
(ii) any transformed equation of a given equation with non-zero coefficient may have zero coefficients but these new coefficients must satisfy (i);
(iii) all the coefficients of non-linear algebraic equation must not be unity;
(iv) GRAEFFE.BAS needs modification in the light of (i), (ii) and (iii).
Under these solvability conditions GRAEFFE.BAS software in Loskor [12] has been modified and subroutine 7 has been developed and extended the software identify the polynomials which are not solvable by this software.

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